Noise and Competition in Strategic Oligopoly *

Ramdan Dridi† and Laurent Germain‡

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Abstract

Focusing on homogeneous beliefs, we can distinguish two commonly shared ideas that, i) the competition between informed traders destroys their trading profits, ii) trading with a noisy signal brings about a loss in the expected profits. So far, it has been proved in the latter framework, that when N strategic and perfectly informed traders compete in the financial market, i) the informativeness of prices increases with the degree of competition and, ii) the aggregate and individual profits go to 0 when N is large. In this paper, we propose a general study where N strategic informed agents have heterogeneous signals. We prove the existence and uniqueness of a linear equilibrium generalizing Kyle (1985) results to the case of N informed traders with heterogeneous beliefs - as Back Cao and Willard (2000) in continuous time. In this general framework, we derive the following striking results: for certain set of information and numbers of competitors in excess to four, i) each individual expected profit is greater than the one obtained in the perfectly informed (and homogeneous beliefs) case. This means that in this context the information has a negative value; ii) the aggregate profit has a finite (strictly) positive limit when N is large. This clearly contradicts any Cournot type results in the context of imperfect competition. iii) Even when an infinite number of insiders compete in the market, the price is no longer efficient and does not fully reveal the final liquidation value of the risky asset. iv) In the particular case where each informed agent is endowed with a signal the precision of which is the same, a) we show that there exists an optimal level of noise in the signals for which each individual expected profit is maximized; b) we show that there exists an optimal size of the market for which the aggregate expected profit is maximized; c) the liquidity is an increasing function of the number of informed traders but has a finite limit for large N; d) the informativeness of prices is a decreasing function of the number of informed traders.

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†Ecole Supérieure de Commerce de Toulouse, Groupe de Finance, 20 boulevard Lascrosses, BP 7010, 31068 Toulouse cedex 7, France. Tel 33 5 61 29 49 43, Fax: 33 5 61 23 32 79, E-mail: r.dridi@esc-toulouse.fr

‡Corresponding author. Ecole Supérieure de Commerce de Toulouse, Groupe de Finance, SUPAERO and Europlace Institute of Finance, Toulouse Business School 20 boulevard Lascrosses, BP 7010, 31068 Toulouse cedex 7, France. Tel 33 5 61 29 49 43, Fax: 33 5 61 23 32 79, E-mail: l.germain@esc-toulouse.fr
1 Introduction

How much information is reflected into prices? How do prices transmit and aggregate information? In the case of competitive traders, Grossman (1976) shows that the equilibrium price aggregates the dispersed private information. If there is no noise in the market - or more generally no sources of uncertainty - prices fully reveal the private information dispersed among the traders (see Grossman and Stiglitz (1980)). Those results are different when considering the case of strategic traders who are aware of their impact into prices. In the case of a monopolistic perfectly informed risk neutral trader Kyle (1985) shows that the price is not fully efficient and only half of the insider’s private information is revealed. In fact, at the equilibrium, the monopoly keeps his informational advantage by optimally camouflaging his information behind noise trading. Those results are different when there are many strategic traders who compete in the market. The competition diminishes the insiders’ informational advantage. Kyle (1984) and Admati and Pfleiderer (1988a) put forward that the informativeness of prices is an increasing function of the degree of competition between the traders. Indeed, prices are all the more informative than the competition between the traders is fierce. In particular, when a large number of strategic perfectly informed traders compete in the market the price is strongly efficient. In the same line but in a dynamic setting Holden and Subrahmanyam (1992,1995) show that the speed of revelation of information increases with the number of competitors.

Another strand of the literature investigates the effects of noise onto prices. There are different sources of noise. The noise can stem from the presence of noise traders as in Kyle (1985) or other sources of uncertainty like endowment shocks when the agents are risk averse (see Biais, Martimort and Rochet (2000)), or a random supply (see Grossman and Stiglitz (1980)) or from noisy signals as in Admati and Pfleiderer (1988b). In a standard linear/normal framework we know that, the more precise is the available information the more profits made by the informed monopoly. ¹ In other words, it is not optimal to add noise into prices or alternatively to make noisier a perfect signal.

The conclusions we can draw from those financial studies is that:

1. On the one hand, noisy information diminishes the informativeness of prices,

2. and, on the other hand, the competition increases the informativeness of prices.

Therefore we could ask the following questions. Could it be that any trade-off arise? Is the competition effect sufficiently strong that, whatever the level of noise in their information set the competitors always bear the cost of the noise in their signals? If so, for which information

¹When the distribution of the risky asset is bimodal or has a three points support, Biais and Germain (2002) and Germain (2002) show that perfectly informed traders would be better off adding noise in their signals ex-ante if they can commit to do so.
sets in the market is it true? \(^2\) Are there optimal values of noise in this general framework? \(^3\) To address those issues in the present paper, we look at a situation where \(N\) strategic informed agents who have heterogeneous beliefs (i.e. are endowed with noisy information \(\tilde{S}_i = \tilde{v} + \tilde{\varepsilon}_i\) \(i = 1, ..., N\) where \(\tilde{v}\) and \(\tilde{\varepsilon}_i\) are normally distributed) compete à la Nash. \(^4\) We will focus on the symmetric case - i.e. all variances of the signals are equal. This has been briefly studied by Admati and Pfleiderer (1988b). In the same vein, Foster and Viswanathan (1996) and Back, Cao and Willard (2000) have studied the impact of correlations on imperfect competition while focusing on heterogeneous beliefs. The main difficulty arising in this type of modelization is that agents have to “forecast the forecasts of others”. In such context, we will derive new results.

So far it is just proved that when \(N\) strategic traders have homogeneous beliefs \(^5\) and compete in the financial market i) the informativeness of prices increases with the degree of competition and ii) the aggregate and individual profits go to 0 when \(N\) is large (See Admati and Pfleiderer (1988a)). As a matter of fact the same authors rightly notice, that “it is straightforward to show that with risk neutral traders, the total profits of the informed traders, \(n\pi(n)\) are decreasing in \(n\). This is analogous to the result that industry profits are decreasing in the number of firms in a Cournot oligopoly”. We stress in this paper that, in the heterogeneous beliefs framework, strategic competition should be rather parameterized by a \(N\)-dimensional vector composed by the precision of each insider’s signal. Indeed, the classical Cournot result holds only for some information structures (in particular for homogeneous beliefs). In the heterogeneous beliefs framework, we show that competition can be diminished by the presence of noise in the information structure. In a different set-up and a dynamic model Foster and Viswanathan (1996) and Back, Cao and Willard (2000) show that the competition may depend on the correlation between the traders’ signals. However, in these models it is assumed that the signals aggregate to a sufficient statistic of the final liquidation \(v\) : \[\frac{1}{N} \sum_{i=1}^{N} s_i = v\] which is not the case in our framework.

Moreover we emphasize the fact that two levels of heterogeneity have to be distinguished in our financial study. In our set up heterogeneous beliefs can stem

- either from different signals which have the same identically distributed distribution (symmetric case), \(^6\)

- or from different probability distributions - in this case the variances of the signals are

\(^2\)When the distribution of the risky asset is bimodal Germain (2002) shows how \(N\) strategic indirect sellers of information can commit to add the optimal ex-ante level of noise in their signals. The commitment is made credible through contracts for the sale of information.

\(^3\)Hellwig (1980) studies in a different set-up (price taking behavior) the aggregation property in a large market.

\(^4\)In the standard literature, the previous questions are often solved within the homogeneous beliefs paradigm. This is for instance the case of Admati and Pfleiderer (1988a), Foster and Viswanathan (1994) and Holden and Subrahmanyam (1992).

\(^5\)The simplest and widespread example corresponds to the case where traders are perfectly informed.

\(^6\)This corresponds to what has been mainly studied.
different.

We will proceed in this paper as follows: first we develop the theory in the latter generic case and for analytical tractability the results are derived for the former symmetric case. This allows us to develop the theory in more details.

More precisely, we derive the following striking results: for certain set of information and a number of competitors $N \geq 4$,

- each individual expected profit is greater than the one obtained in the perfectly informed case. This means that in this context the information has a negative value;

- the aggregate profit has a finite (strictly) positive limit when $N$ is large. This clearly contradicts any Cournot type results in the context of imperfect competition. It is also surprising to notice that in this context if the information gathering were costly, a larger number of informed traders than in the perfect informed case could recover the cost. This shows that in some cases, with imperfect competition, the Grossman and Stiglitz paradox is less severe.

- Even when an infinite number of insiders compete in the market the price is no longer efficient and does not fully reveal the final value of liquidation of the risky asset. In other words, with heterogeneous beliefs the collection of signals do not aggregate to the true value of the asset.

- In the particular case where each heterogeneous informed agent is endowed with a signal the whose precision is the same for each agent we show that there exists an optimal level of noise in the signals for which each individual profit is maximized. Moreover there exists a range of the form $[0, B_N]$ with $B_N = \frac{\sigma \sqrt{N + 1} \sqrt{N - 3}}{2}$, for which noisy signals are pareto preferable in this economy. Thus the results is not reduced to a single point the optimal level of noise in the signals.

- For each given level of precision of the signals, there exists an optimal size of the market $N^*(\sigma)$ for which the aggregate expected profit is maximized. This entails that a seller of information endowed with a signal the precision of which is $\sigma$ would transmit his information to $N^*(\sigma)$ traders in order to maximize the proceeds of the sale.

- The liquidity is not always an increasing function of the number of informed traders and may have a finite limit for large $N$.

- The informativeness of prices is not always increasing with the number of informed traders.

Furthermore, we address another issue. In the framework of linear equilibria modeling with normality and when the traders have heterogeneous beliefs, two related issues remain unsolved.
i) Do we still have the uniqueness of a particular linear equilibrium (linear in prices as well as in quantities)? ii) More generally, do we have the uniqueness property in the larger class of linear equilibrium and where the linearity solely refers to the pricing schedule?

As a matter of fact, Kyle (1984) and Admati and Pfleiderer (1988a) have studied the case where N informed traders endowed with the same signal $S$ compete in the market and have shown the existence of a linear equilibrium. Admati and Pfleiderer (1988b) Foster and Viswanathan (1996) and Vives (1995) have shown also the existence of a linear equilibrium when traders have heterogeneous beliefs. In this respect, our results are the following. We first show the existence and the uniqueness of a linear equilibrium. This generalizes the result derived by Kyle (1985) in the case of a single perfectly informed monopoly to the case where N strategic informed traders are endowed with different signals. This relies to the best of our knowledge on new proofs since so far what has been exhibited is the existence and not the uniqueness of the particular linear equilibrium. In effect, while in Kyle (1985), Admati and Pfleiderer (1988a) and Holden and Subrahmanyam (1992), each informed trader knows at the equilibrium the quantity submitted by each other one, in our framework there is still a persisting uncertainty due to the noisy signals and therefore the informed noisy trades.\footnote{This uncertainty is reinforced in the general case where insiders signals are distributed according to non identical probability distributions.} In a continuous time framework Back, Cao and Willard (2002) show the existence and the uniqueness of the equilibrium with imperfect competition.

The paper is organized as follows. We first lay out in section 2 the general set up and the considered model generalizing Kyle (1985). We show in section 3 the existence and uniqueness of a linear equilibrium and characterize the equilibrium as well as the expected profits performed by each informed agent in the heterogeneous beliefs context. In section 4, we provide the general study of the equilibrium properties and delineate the regions for which each informed trader is better off with respect to the perfectly informed case. We discuss in section 5 the particular situation in which each informed heterogeneous agent is endowed with a signal of which precision is the same. This allows us to derive analytical results. We show the existence of an optimal level of noise in the signals for each given number of insiders as well as the optimal size of the market for each given level of precision. We then focus on the liquidity, individual and aggregate profits and informativeness properties at equilibrium. Finally section 6 states some concluding remarks.

\section{The set up}

We consider a financial market with a risky asset normally distributed $\tilde{v} \sim \mathcal{N}(0, \sigma_v^2)$. We denote the final liquidation value $v$. There are three types of agents:

1. $N$ risk neutral informed traders who observe in advance a signal $\tilde{S}_i = \tilde{v} + \tilde{\varepsilon}_i$, where $\tilde{\varepsilon}_i$ is a random disturbance term (the noise) and we will assume that:

$$\tilde{\varepsilon}_i \sim \mathcal{N}(0, \sigma^2_i),$$

$$\{\tilde{v}, \tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_N\}$$ are mutually independent.

2. Liquidity traders who submit market orders $\tilde{u} \sim \mathcal{N}(0, \sigma^2_u)$ and such that $\{\tilde{v}, \tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_N, \tilde{u}\}$ are mutually independent.

3. Risk neutral market makers, who observe the aggregate volume $\tilde{w}$ and set rationally the price in a Bayesian way.

Note that the independence assumption on $\{\tilde{v}, \tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_N\}$ does not mean that the signals $\{\tilde{S}_1, \ldots, \tilde{S}_N\}$ are independent. Indeed, there is a correlation due to the common information structure $\tilde{v}$.

A strategy for the informed agent $i = 1, \ldots, N$ is a Lebesgue measurable function: $X_i : \mathbb{R} \rightarrow \mathbb{R}$, determining his market order as a function of the observed signal $S_i$. For given strategies $(X_1, \ldots, X_N)$, let $\tilde{x}_i = X_i(\tilde{S}_i)$. These strategies determine the aggregate order flow

$$\tilde{w} = \sum_{i=1}^{N} \tilde{x}_i + \tilde{u}.$$

Market makers observe the realization of the order flow, but not any of its components, and are engaged in a competitive auction. The outcome of this competition is described by a Lebesgue measurable function: $P : \mathbb{R} \rightarrow \mathbb{R}$.

Given $(P, X_1, \ldots, X_N)$, we denote $\tilde{p} = P(\tilde{w})$ and let $\tilde{\pi}_i = (\tilde{v} - \tilde{p}) \tilde{x}_i$ the resulting trading profit of each insider $i = 1, \ldots, N$.

The equilibrium conditions are that the competition in which market makers are involved drives their expected profits to zero conditional on the aggregate submitted volume and that the informed traders choose their trading strategies so as to maximize their expected profits.

**Definition 2.1**: $(P, X_1, \ldots, X_N) \in L_2^{N+1}$ is an equilibrium if:

$$E[\tilde{v} - \tilde{p}/\tilde{w}] = 0,$$

and for all $X_i \in L_2$, given the (rational) beliefs of the market makers, and the corresponding price function $(P(.))$, each informed chooses $X_i$ to maximize his expected profits:

$$\tilde{x}_i \in \arg\max_{x \in \mathbb{R}} E \left[ \left( \tilde{v} - P(x + \sum_{j \neq i} \tilde{x}_j + \tilde{u}) \right) x/S_i \right].$$

$L_2$ corresponds to the set of square integrable Lebesgue measurable functions.
Definition 2.2 : \((P, X_1, \ldots, X_N) \in L_2^{N+1}\) is a linear equilibrium if in addition there exists a scalar \(\lambda \in \mathbb{R}_+\):
\[
\ddot{p} = E[\ddot{v}/\ddot{w}] = \lambda \ddot{\omega}. \tag{2.3}
\]

We derive the unique perfect Bayesian linear equilibrium of this game.

### 3 Equilibrium Existence and Uniqueness

We first start by stating the result in the monopoly case which is distinct from the oligopoly case \((N \geq 2)\). We characterize the linear equilibrium in the case of a monopolistic imperfectly informed trader who observes a signal \(\ddot{S} = \ddot{v} + \ddot{\varepsilon}\). This signal corresponds to the previous oligopoly description where \(N=1\).

**Proposition 3.1** : In the monopoly case \((N = 1)\), there exists a unique linear equilibrium defined by \(\ddot{x} = \beta^*(\sigma)\ddot{S}\) and \(\ddot{p} = \lambda^*(\sigma)\ddot{\omega}\), where \(\sigma^2 = \text{Var}(\ddot{\varepsilon})\), such that:
\[
\beta^*(\sigma) = \frac{\sigma_u}{\sigma_v} \frac{1}{\sqrt{1 + \tau}},
\]
\[
\frac{1}{\lambda^*(\sigma)} = 2 \frac{\sigma_u}{\sigma_v} \sqrt{1 + \tau}, \tag{3.1}
\]
\[
\tau = \frac{\sigma^2}{\sigma_v^2}.
\]

In equilibrium the expected profit \(\pi^*(\sigma)\) is:
\[
\pi^*(\sigma) = \frac{1}{2} \frac{\sigma_u \sigma_v}{\sqrt{1 + \tau}} \frac{1}{\sqrt{1 + \tau}}, \tag{3.2}
\]
\[
\tau = \frac{\sigma^2}{\sigma_v^2}.
\]

**Proof** : See appendix A.1.

Figure 1 represents the individual monopolistic reaction as a function of the noise in the signal and where the normalization is taken with respect to the perfectly informed case. We observe that the more accurate the information is the more reactive the monopoly. As usual this illustrates the confidence of the informed agent to his private information since the inverse of the variance of the disturbance term represents the precision of the information.
Figure 1: Individual Reaction (N=1).

Figure 2 represents the liquidity in the monopolistic case as a function of the noise in the signal and where the normalization is taken with respect to the perfectly informed case. We observe that the more accurate the information is the less liquid the market. Again, this is a standard result. At the limit, when there is no noise either stemming from noise trading or from the noise in the informed trader’s signal, there is no liquidity in the market.

Figure 2: Liquidity (N=1).

Figure 3 represents the expected profit achieved by the monopoly as a function of the noise in the signal and where the normalization is taken with respect to the perfectly informed case. Given that the expected profit is equal to the inverse of the liquidity $\lambda^*(\sigma)$ times the variance of the noise trading $\sigma^2_u$, we have that the expected profit is a decreasing function of the noise in the signal and at the limit is equal to zero. This illustrates that in this linear setting a monopoly is
always worse-off gathering a noisier information. As a consequence, it is never optimal ex-ante to add noise to his own information.

We have shown the standard and widely accepted results in case of a single informed agent. However in case of imperfect competition, those results do no longer hold.

**Proposition 3.2:** There exists a unique linear equilibrium defined by $i = 1, \ldots, N$, $\bar{x}_i = \beta_i^*(\sigma) \bar{S}_i$ and $\bar{p} = \lambda^*(\sigma) \bar{\omega}$ and where $(\beta_1^*, \ldots, \beta_N^*, \lambda^*) (\sigma)$ is given by:

$$
\beta_i^*(\sigma) = \frac{\sigma_u}{\sigma_v} \left\{ \frac{1}{\sum_{j=1}^{N} \frac{1 + \tau_j}{(1 + 2\tau_j)^2}} \right\}^{\frac{1}{2}} \frac{1}{1 + 2\tau_i}, \quad i = 1, \ldots, N,
$$

$$
\frac{1}{\lambda^*(\sigma)} = \frac{\sigma_u}{\sigma_v} \left\{ \frac{1}{\sum_{j=1}^{N} \frac{1 + \tau_j}{(1 + 2\tau_j)^2}} \right\}^{\frac{1}{2}} \left[ \frac{1}{1 + \sum_{j=1}^{N} \frac{1}{1 + 2\tau_j}} \right], \quad \text{with } \tau_j = \frac{\sigma_j^2}{\sigma_v^2}, j = 1, \ldots, N.
$$

In equilibrium the expected profit $\pi_i^*(\sigma)$ for the agent $i$ is:

$$
\pi_i^*(\sigma) = \frac{\sigma_u \sigma_v}{(1 + 2\tau_i)^2} \left\{ \frac{\sum_{j=1}^{N} \frac{1 + \tau_j}{(1 + 2\tau_j)^2}}{\left\{ \frac{1}{1 + \sum_{j=1}^{N} \frac{1}{1 + 2\tau_j}} \right\}^{\frac{1}{2}}} \right\}, \quad i = 1, \ldots, N,
$$

with $\tau_i = \frac{\sigma_i^2}{\sigma_v^2}$.

For $\sigma = 0$, we obtain the benchmark:

$$
\pi_i^*(0, N) = \frac{\sigma_u \sigma_v}{(N + 1) \sqrt{N}}.
$$
The expected aggregate profit \( \pi^*(\sigma) \) is, at equilibrium, given by:

\[
\pi^*(\sigma) = \sum_{i=1}^{N} \pi_i^*(\sigma) = \frac{\sigma_v \sigma_u \left\{ \sum_{j=1}^{N} \frac{1 + \tau_j}{(1 + 2\tau_j)^2} \right\}^{\frac{1}{2}}}{1 + \sum_{j=1}^{N} \frac{1}{1 + 2\tau_j}},
\]

(3.6)

with \( \tau_j = \frac{\sigma_j^2}{\sigma_v^2}, \ j = 1, \ldots, N. \)

For \( \sigma = 0 \), we have \( \pi^*(0, N) = \frac{\sigma_v \sigma_u \sqrt{N}}{N + 1} \).

**Proof**: See appendix A.2.

To the best of our knowledge, this result generalizes all the existing ones in the literature. More precisely, Kyle (1984), Admati and Pfleiderer (1988b) show the existence of a linear equilibrium while assuming that, i) the pricing rule is linear, ii) each informed agent conjectures that each other informed trader’s strategy is linear in his signal (and consequently that in equilibrium informed traders strategies are linear). This does not prove in any case, i) the uniqueness of the linear equilibrium \(^9\), ii) that the class of possible linear equilibria corresponds to linear reaction of each informed agent in their signal.

In light of this, proposition 3.2 states that, the class of linear equilibria is indeed reduced to a singleton and that the reaction of each informed agent is, at the equilibrium, linear.

In the next proposition we determine the expected profits of the traders at the individual and aggregated level.

Moreover it is worth noticing that each individual expected informed traders profit appear as \( \frac{1}{2} \sigma_v \sigma_u \), the expected profit achieved by the monopolistic trader in case of perfect information times a non linear function of the collection of noise in the signal. Indeed this non linearity shed some new lights on strategic imperfect competition.

### 4 Equilibrium Properties

**Proposition 4.1**: In equilibrium, the individual reaction \( \beta_i^*(\sigma) \) of the agent \( i \) is decreasing with each other competitor’s precision of information; that is to say, increasing with \( \sigma_j, \ j \neq i, \) and increasing with its own information precision, that is to say, decreasing with \( \sigma_i. \)

In equilibrium, for \( i = 1, \ldots, N, \) the individual profit \( \pi_i^*(\sigma) \) is an increasing function of \( \sigma_{-i} \) the

\(^9\)We emphasize here that linear equilibrium refers to the linearity in price according to definition 2.2 and not to linearity in informed traders strategies.
\((N-1)\) vector of noise in the \(-i\) signals in the economy (and a decreasing function in the noise in the signal \(\sigma_i\)).

**Proof**: See appendix B.1.

This proposition shows that the informed traders react all the more to their private information than the other traders’ information is noisy. Indeed, since the total amount of noise in the other informed agents signals increases the traders can camouflage more their information. And as a consequence the individual profits increases with the noise in the other competitors signals.

**Proposition 4.2**: For \(i = 1, \ldots, N\), we define:

\[
b(\tau_{-i}) = 1 - \sum_{j \neq i} \frac{3 + 2\tau_j}{(1 + 2\tau_j)^2},
\]

\[
c(\tau_{-i}) = 2 + \sum_{j \neq i} \frac{2\tau_j - 1}{(1 + 2\tau_j)^2},
\]

(4.1)

then:

- if \(b(\tau_{-i}) \geq 0\), the liquidity \(\lambda^{*-1}(\tau)\) (or \(\lambda^{*-1}(\sigma)\)) is, in equilibrium, an increasing function of \(\tau_i = \frac{\sigma_i^2}{\sigma_v^2}\) for any fixed \(\tau_{-i}\),

- if \(b(\tau_{-i}) \leq 0\) and \(c(\tau_{-i}) \leq 0\), the liquidity \(\lambda^{*-1}(\tau)\) (or \(\lambda^{*-1}(\sigma)\)) is, in equilibrium, a decreasing function of \(\tau_i = \frac{\sigma_i^2}{\sigma_v^2}\) for any fixed \(\tau_{-i}\),

- if \(b(\tau_{-i}) \leq 0\) and \(c(\tau_{-i}) \geq 0\), then:
  
  * for \(\tau_i \leq -\frac{c(\tau_{-i})}{2b(\tau_{-i})}\), the liquidity \(\lambda^{*-1}(\tau)\) (or \(\lambda^{*-1}(\sigma)\)) is, in equilibrium, an increasing function of \(\tau_i = \frac{\sigma_i^2}{\sigma_v^2}\) for any fixed \(\tau_{-i}\),

  * for \(\tau_i \geq -\frac{c(\tau_{-i})}{2b(\tau_{-i})}\), the liquidity \(\lambda^{*-1}(\tau)\) (or \(\lambda^{*-1}(\sigma)\)) is, at equilibrium, a decreasing function of \(\tau_i = \frac{\sigma_i^2}{\sigma_v^2}\) for any fixed \(\tau_{-i}\),

**Remark 4.1**: All the previous cases are possible.

**Proof**: See appendix B.2.
Given that the aggregate profit is equal to the inverse of the liquidity times the variance of the noise trading $\sigma_{u}^{2}$, the interpretations of propositions 4.2 are as follows. In our framework, the liquidity in the market and the aggregate expected profit are not monotonic functions of the set of information in the market. In particular,

- if there is a sufficient amount of noise in the market $b(\tau_{-i}) \geq 0$, then the more precise is the collected information by an insider the smaller the depth and the bigger the aggregate expected profit. In other words, both price pressure and aggregate expected profit increase with the precision of the information collected by one trader all things being equal.

- if the amount of noise in the market is small enough (but not too small) $b(\tau_{-i}) \leq 0$ and $c(\tau_{-i}) \geq 0$ - as long as the collected information is relevant - $(\tau_{i} \leq -\frac{c(\tau_{-i})}{2b(\tau_{-i})})$ - the price pressure and the aggregate expected profit increase with the precision of the information collected by one trader all things being equal. If not - $\tau_{i} \geq -\frac{c(\tau_{-i})}{2b(\tau_{-i})}$ - the price pressure and the aggregate expected profit decrease with the precision of the information collected by one trader all things being equal.

- For small levels of noise in the signals $b(\tau_{-i}) \leq 0$ and $c(\tau_{-i}) \leq 0$ - the price pressure and the aggregate expected profit decrease with the precision of the information collected by one trader all things being equal.

To summarize the previous effects, a small amount of noise in the market allows to sufficiently soften the competition. Indeed, in the latter case, the competition effect supersedes the cost of noise in the signal. On the contrary a larger amount of noise in the market does not enable to sufficiently soften the competition. This latter effect is smaller than the cost of noise in the insider’s signal. Finally, for intermediate amount of noise in the market, we observe different regions for which one effect dominates the other one.

These new patterns- namely the non monotonicity of both liquidity and expected profits- leads us to characterize sets of information for which noisy information is pareto preferable.

**Proposition 4.3 :** Let $N$ informed agents endowed with signals $\tilde{S}_{i} = \tilde{v} + \tilde{\epsilon}_{i}$ with $\{\tilde{\epsilon}_{1}, \ldots, \tilde{\epsilon}_{N}, \tilde{v}, \tilde{u}\}$ mutually independent and where $\tilde{u}$ corresponds to the noise trading, $\tilde{v}$ the liquidation value of the risky asset and $\tilde{x}_{i} = \beta_{i} \ast (\sigma) \tilde{S}_{i}$ the orders submitted by each competing informed traders in equilibrium, then if we define $\Sigma_{N}$ as:

$$
\Sigma_{N} = \left\{ \sigma = (\sigma_{1}, \ldots, \sigma_{N})' \in \mathbb{R}_{+}^{N} \right\},
$$

$$
\forall \sigma \in \Sigma_{N}, \forall i = 1, \ldots, N,
$$

$$
\pi^{*}_{i}(\sigma_{1}, \ldots, \sigma_{N}) \geq \pi^{*}_{i}(0, \ldots, 0) = \pi^{*}_{i}(0, N),
$$

$$
\exists i_{o} \in \{1, \ldots, N\} / \pi^{*}_{i_{o}}(\sigma_{1}, \ldots, \sigma_{N}) > \pi^{*}_{i_{o}}(0, \ldots, 0).
$$
We have:

- for $N \geq 4$, $\Sigma_N$ is non-empty. In other words, noise in the collection of signals is ex-ante optimal.
- For $N \leq 3$, $\Sigma_N$ is empty.

Proof: See appendix B.3.

This corresponds in our opinion to one of the most important result of the paper. Indeed, it highlights the fact that collecting information can be suboptimal in a competitive market as soon as the competition is strong enough ($N \geq 4$). In effect, it is clearly seen from proposition 4.3 that the information has a negative value. The implication of the latter results is that for a given structure of information (with strategic informed agents in case of imperfect competition) that can either be a bank with subdivisions or a direct seller of information like in Admati and Pfleiderer (1986-1990), it is always worse-off to provide each competitor with the correct valuation of the risky project. This is in the line of Admati and Pfleiderer (1986) for competitive agents and Germain (1998) for indirect sellers of information. The level of competition measured by $N \geq 4$ is tightly related to the assumptions made on the model namely the normality and independence assumptions made on both the asset, the noise trading and the signals.

5 Symmetric Case

We focus in this section on the symmetric heterogeneous beliefs case that is when all informed agents are endowed with signals of same precision, i.e., $\sigma_i = \sigma$ for $i = 1, \ldots, N$.

We still maintain the mutual independence of $\{\tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_N, \tilde{v}, \tilde{u}\}$. Then we can state as corollaries of the previous propositions 3.2 – 4.2 that:

**There exists a unique perfect Bayesian linear equilibrium defined as in proposition 3.2. In this particular case $(\beta^*_1, \ldots, \beta^*_N, \lambda^*) (\sigma, N)$ is characterized by:**

\[
\beta^*_i(\sigma, N) = \beta^*(\sigma, N) = \frac{\sigma_u}{\sigma_v} \frac{1}{\sqrt{N} \sqrt{1 + \frac{\sigma^2}{\sigma^*_v}}}, \quad i = 1, \ldots, N,
\]

\[
\frac{1}{\lambda^*(\sigma, N)} = \frac{\sigma_u}{\sigma_v} \frac{N + 2\frac{\sigma^2}{\sigma^*_v}}{\sqrt{N} \sqrt{1 + \frac{\sigma^2}{\sigma^*_v}}}
\]

(5.1)
In equilibrium, the individual profit $\pi_i^*(\sigma, N)$, for $i = 1, \ldots, N$ is given by:

$$\pi_i^*(\sigma, N) = \sigma_v \sigma_u \left( 1 + \frac{\sigma^2}{\sigma_v^2} \right)^{1/2} \sqrt{N \left( N + 1 + \frac{2\sigma^2}{\sigma_v^2} \right)}$$

(5.2)

and the aggregate profit $\pi^*(\sigma, N)$ is:

$$\pi^*(\sigma, N) = \sigma_v \sigma_u \left( 1 + \frac{\sigma^2}{\sigma_v^2} \right)^{1/2} \frac{N}{N + 1 + \frac{2\sigma^2}{\sigma_v^2}}$$

(5.3)

Figures 4 and 5 represent, in the symmetric case, the liquidity $\frac{1}{\lambda^*}(\sigma, N)$ as a function of both the number of insiders and the level of noise in the signals:

- First, for a fixed positive level of noise in the signals $\sigma > 0$, the liquidity is first decreasing with the number of insiders reaching a minimum at the value $N = 1 + \frac{2\sigma^2}{\sigma_v^2}$ along the given hyperplane and then increasing with the number of insiders. This seriously differs from the case of perfect information $\sigma = 0$ where the liquidity is always increasing with the number of informed traders.

- Second, for a small number of insiders ($N \leq 3$), we observe as usual, that the liquidity is an increasing function of the level of noise in the signals for a fixed number of informed traders.

\[ 10 \]

Of course, this implicitly assumes that we are extending the variables $N$ to the real axis. For sake of clarity, we will not discuss here in further details the fact that in practice $N \in \mathbb{N}$ the set of positive integers. Indeed one has rather to take the integer part and to discuss whether it is greater than one. See however proposition 5.3 for the exact statements.
Third, for a larger number of informed traders \((N \geq 4)\), and any given number of insiders, the liquidity is first decreasing with the level of noise in the signals reaching a minimum at the value \(\sigma = \sigma^*(N) = \sqrt{\frac{N - 3}{2}} \sigma_v\) along the given hyperplan and then increasing with the level of noise in the signals. In this case, we observe a “basin”.

\[\begin{align*}
\text{Figure 5: Liquidity } N \geq 4. \\
\text{Figure 6: Individual Profit } N < 4.
\end{align*}\]

Figures 6 and 7 represent, in the symmetric case, the individual profit achieved by each informed agent as a function of both the level of noise in the signals and the number of insiders. For any fixed level of noise in the signals, the individual profit is always decreasing with the number of insiders.
• For a small number of informed agents \((N \leq 3)\), the individual profit is always decreasing with the level of noise in the signals for any fixed number of insiders.

• For a large number of informed traders \((N \geq 4)\), and any given number \(N\) of insiders, the individual profit is first increasing with the noise in the signals reaching a maximum at the value \(\sigma = \sigma^*(N) = \sqrt{\frac{N - 3}{2}}\sigma_v\) and then decreasing with the level of noise in the signals.

This illustrates the idea that noise in the signals has basically two effects, one costly which is due to the fact that, in some sense and for a part of his trade, the insider is behaving as a noise trader; and the other beneficial since it tends to weaken the competition. In other words, for a large number of insiders the beneficial effect of noise in the signals supersedes the costly one.
Figures 8 and 9 represent, in the symmetric case, the normalized individual profit achieved by each informed agent as a function of both the level of noise in the signals and the number of insiders. We have chosen the normalization with respect to the profit $\pi^*_i(0,N)$ performed in case of perfect information $\sigma = 0$. In other words, we have reported the ratio $\frac{\pi^*_i(\sigma,N)}{\pi^*_i(0,N)}$ as a function of the number of insiders and the level of noise in the signals. Again, two cases have to be envisioned.

- First, for a small number of insiders ($N \leq 3$), this ratio is decreasing with the level of noise in the signals and increasing with the number of informed traders. However for a small number of insiders, this ratio is always smaller than one.

- Second, for a larger number of insiders ($N \geq 4$), there exist for each given number of insiders, levels of noise in the signals for which this ratio is bigger than one. Moreover, it is increasing with the number of informed traders, for a fixed level of noise in the signals. For a given number of insiders, the ratio is first increasing reaching a maximum at the value $\sigma = \sigma^*(N) = \sqrt{\frac{N - 3}{2} \sigma_v}$ in the given hyperplan $(N)$ and then decreasing with the level of noise in the signals.

![Normalized Individual Profit](image)

Figure 9: Normalized Individual Profit $N \geq 4$.

Figures 10 and 11 represent the normalized aggregate profit as a function of the level of noise in the signals and the number of insiders. The normalization is performed with respect to the profit achieved by the monopoly in Kyle (1985) setup ($\pi^*(0,1) = \frac{1}{2} \sigma_u \sigma_v$).
For any fixed positive level of noise in the signals $\sigma > 0$, the aggregate profit is first increasing with the number of insiders to reach a maximum in the given hyperplan at the value $N = 1 + \frac{2\sigma^2}{\sigma_v^2}$. Again, this result has to be compared with the perfect information case where the aggregate profit is always decreasing with the number of insiders.

For a small and fixed number of informed traders ($N \leq 3$), the aggregate profit is always decreasing with the level of noise in the signals.

For a larger and given number of informed traders ($N \geq 4$), the aggregate profit is first increasing with the level of noise in the signals to reach a maximum at the value $\sigma = \sigma^*(N) = \sqrt{\frac{N - 3}{2}} \sigma_v$ and then decreasing with the level of noise in the signals. Again, this has to be related to the two effects (beneficial and costly ones) of the noise in the collection of information.

**Proposition 5.1**: For all $N \geq 4$, and for all $\sigma \in \left(0, \frac{\sigma_v \sqrt{N + 1}}{2} \sqrt{N - 3}\right)$, we have:

$$\forall i = 1, \ldots, N, \quad \pi^*_i(\sigma, N) > \pi^*_i(0, N).$$

In other words, noise in signals is preferable. And furthermore we have an equivalence result:

$$\pi^*_i(\sigma, N) > \pi^*_i(0, N) \iff \sigma \in \left(0, \frac{\sigma_v \sqrt{N + 1}}{2} \sqrt{N - 3}\right).$$

11 The same remarks as for the liquidity apply, namely, the fact that $1 + \frac{2\sigma^2}{\sigma_v^2}$ is not necessarily an integer. See however proposition 5.3 for the exact statements.
Moreover there exists an optimal level of noise in the signals, for $N \geq 4$:

$$\sigma^*(N) = \sigma_v \sqrt{\frac{N - 3}{2}},$$

which maximizes each individual profit in the symmetric case.

In equilibrium for the symmetric case, the aggregate profits for each player, when evaluated at the optimal level of noise are given by:

- $\pi^*_{informed\ traders}(N) = \pi^*(N) = \frac{\sigma_v \sigma_u}{2 \sqrt{2}} \frac{1}{\sqrt{1 - \frac{1}{N}}}$,

- $\pi^*_{noise\ traders}(N) = -\pi^*(N) = -\frac{\sigma_v \sigma_u}{2 \sqrt{2}} \frac{1}{\sqrt{1 - \frac{1}{N}}}$,

- $\pi^*_{market\ makers} = 0$.

**Proof**: See appendix C.1.

Proposition 5.1 states that for a whole interval $]0, B_N[$ with $B_N = \frac{\sigma_v \sqrt{N + 1} \sqrt{N - 3}}{2}$ noise in signals is preferable. This means that the result is not limited to a single point. Of course there is a particular value $\sigma^{*2}(N)$ for which the expected profits are maximized. Moreover, it is important to notice that while $N$ increases the interval is at the limit the whole real positive line. In other words, the bigger the number of competitors and the easier is the pareto improvement while dealing with noisy information. In this respect, we can argue that the value of the information is more and more negative. Moreover, this result solely relies on the heterogeneous beliefs framework. Indeed, we already know from Admati and Pfleiderer (1988a), that in case of homogeneous beliefs i.e. same signals the result is no longer true.
The expected aggregate profits \( \pi^*(N) = N \pi_i^*(N) \) do not go to zero when \( N \) is large more precisely it tends to a finite positive value \( \frac{\sigma_v \sigma_u}{2\sqrt{2}} \). This clearly contradicts the common idea and intuition based on the standard Cournot result. It is important to notice that the standard literature like Admati and Pfleiderer (1988a) has focused on homogeneous beliefs for which the Cournot result still holds. In our framework of heterogeneous beliefs two cases have indeed to be distinguished. In the former case (when the precision of the information is fixed) the Cournot result also holds and the expected aggregate profits go to zero with the number of informed traders. In the latter case as soon as \( \sigma^2 N = O(N) \) the expected aggregate profits go to a finite positive limit. This condition is fulfilled for the optimal level of noise in the signals \( \sigma^*(N) \).

In order to deeply understand this striking result, we need to focus on the informativeness of prices and also on the process of aggregation of the collection of information. This is the purpose of the next proposition.

![Figure 12: Optimal Level of Noise.](image)

In figure 12, we have plotted the optimal level of noise in the signals \( \frac{\sigma^*(N)}{\sigma_v^2} \) as a function of the number of informed agents. It is an affine function of the number of insiders. Therefore, when \( N \) is large, the optimal noise in the signals is also large.

Figures 13 and 14 represent the liquidity respectively for the optimal level of noise in the signals \( \sigma = \sigma^*(N) \) and the perfect information \( \sigma = 0 \) cases respectively of the number of insiders. In both cases, the liquidity functions are increasing with the number of insiders. However, while in the perfect information case the liquidity tends to infinity with the number of informed agents, in the optimal noise in the signals case, it tends to a finite limit \( \lambda_\infty^{-1} = \frac{\sigma_u}{\sigma_v} 2\sqrt{2} \) when the number of insiders is large.
Figure 13: Liquidity Optimal Symmetric Noise $N \geq 4$.

Figure 14: Liquidity Perfect Information $N \geq 4$. 
Figure 15: Individual Profit: Optimal Symmetric Noise versus Perfect Information $N \geq 4$.

Figure 15 represents the individual profits $\pi_i^*(N)$ (thick line) and $\pi_i^*(0, N)$ (dash line) for the optimal level of noise in the signals $\sigma^*(N)$ and for the perfect information $\sigma = 0$ cases respectively as a function of the number of informed agents. Both individual profits are strictly decreasing functions of the number of insiders and converging towards 0 when the number of informed traders is large enough. However the ratio of the two (see figure 16 ) tends also to infinity when the number of insiders is large enough. This means that the individual profit achieved by each insider for the case of optimal noise in the signals is “infinitely greater” than that observed in the perfect information case when the number of insiders is large.

Figure 16: Ratio Individual Profit: Optimal Symmetric Noise versus Perfect Information $N \geq 4$.
Figure 17: Aggregate Profit: Optimal Symmetric Noise versus Perfect Information $N \geq 4$.

Figure 17 represents the aggregate profits $\pi^*(N)$ (thick line) and $\pi^*(0, N)$ (dash line) for the optimal level of noise in the signals $\sigma^*(N)$ and for the perfect information $\sigma = 0$ cases respectively as a function of the number of informed agents. Both aggregate profits are strictly decreasing functions of the number of insiders. However, while the aggregate profit in the optimal noise in the signals case $\sigma^*(N)$ has a finite strictly positive limit $\frac{\sigma_u \sigma_v}{2\sqrt{2}}$, the aggregate profit converges towards 0 in case of perfect information when the number of informed traders is large enough.

Proposition 5.2: For each level of precision and number of informed traders $(\sigma, N)$, the informativeness of prices $I(\sigma, N)$ is given in equilibrium by:

$$I(\sigma, N) = \frac{1}{\text{Var}(\tilde{v}/\tilde{p})} = \frac{1}{\sigma_v^2} \left( \frac{N + 1 + 2 \frac{\sigma^2}{\sigma_v^2}}{1 + 2 \frac{\sigma^2}{\sigma_v^2}} \right), \quad (5.7)$$

The informativeness of prices $I^*(N)$ is given for the optimal level of noise in the signals $\sigma^*(N)$ and at equilibrium by:

$$I^*(N) = \text{Var}(\tilde{v}/w)^{-1} = \frac{N + 1 + 2 \frac{\sigma^2}{\sigma_v^2}}{N - 2}, \quad (5.8)$$

and when $N$ is large, $I^*(N)$ is equivalent to:

$$I^*(N) = \text{Var}(\tilde{v}/w)^{-1} \sim \frac{2}{\sigma_v^2}. \quad (5.9)$$

If we denote $S = \frac{1}{N} \sum_{i=1}^{N} S_i$, then:

- if $\sigma^2$ is fixed,
  
  $* \frac{S}{N \to +\infty} v$,

  $* \frac{p}{N \to +\infty} v$
\[ \text{if } \sigma^2 = \sigma^*^2(N), \]
\[ * \overline{S} - v \xrightarrow{\mathcal{L}} \frac{\sigma_v}{\sqrt{2}} \overline{X}, \text{ with } \overline{X} \sim \mathcal{N}(0,1), \]
\[ * \tilde{p} - \frac{1}{2}v \xrightarrow{\mathcal{L}} \frac{\sigma_v}{\sqrt{2}} \overline{Y}, \text{ with } \overline{Y} = \frac{1}{\sqrt{2}} \left( \overline{X} + \frac{u}{\sigma_u} \right) \sim \mathcal{N}(0,1), \]
\[ * \tilde{p} - \overline{S} \xrightarrow{\mathcal{L}} \frac{1}{2}v + \frac{\sqrt{3}}{4} \sigma_v \overline{Z}, \text{ with } \overline{Z} = -\frac{1}{\sqrt{3}} \overline{X} + \frac{2}{\sqrt{3}} \frac{u}{\sigma_u} \overline{Y} \sim \mathcal{N}(0,1), \]

**Proof**: See appendix C.2.

Proposition 5.2 shows that the informativeness of prices tends to infinity for a fixed variance whereas there is a finite limit for the case of optimal noise in the signals or more generally when the noise in the signals of the same order as the optimal one. This shows that in the latter case the efficiency is never reached in this market. This is confirmed by the results on the aggregation of information. Indeed our results are of three orders. When the variance is fixed, both the statistic \( \overline{S} \) and the price \( \tilde{p} \) aggregate the actual value \( v \). For the case of optimal noise in the signals, the statistic \( \overline{S} \) appears at the limit as the sum of the actual value \( v \) plus some error term. Thus the collection of signals \( \overline{S} \) does not aggregate the final liquidation value of the risky asset. Nor does the price \( \tilde{p} \) aggregate the value \( v \). Indeed it corresponds to \( \frac{1}{2}v \) plus some error term. Moreover the price \( \tilde{p} \) does not aggregate the statistics \( \overline{S} \).

This clearly corresponds to new results heavily relying on both heterogeneous beliefs and optimal noise in signals or more generally on noise in signals of the same order as the optimal one. In this context, we can understand why at the limit the expected aggregate profits do not go to zero for large \( N \). This is essentially due to the non perfect aggregation of information. As a result, in this economy as the number of competitors becomes larger, the market does not become more efficient.

![Informativeness Optimal Symmetric Noise](image)

**Figure 18**: Informativeness Optimal Symmetric Noise.
Figures 18 and 19 represent the informativeness of prices respectively for the optimal level of noise in the signals $\sigma = \sigma^*(N)$ and for the perfect information $\sigma = 0$ cases respectively as a function of the number of insiders. While in the former case the informativeness of prices is decreasing with the number of insiders and has a finite positive limit $\frac{2}{\sigma_v^2}$, in the latter case it is increasing and tends to infinity with the number of insiders.

![Figure 19: Informativeness Perfect Information.](image)

**Proposition 5.3:** If we define $N^*(\sigma)$ as:

\[
N^*(\sigma) = 1 + 2\frac{\sigma^2}{\sigma_v^2}, \text{ if } 1 + 2\frac{\sigma^2}{\sigma_v^2} \in \mathbb{N},
\]

\[
N^*(\sigma) = \arg\max_{N \in (\tilde{N}_1(\sigma), \tilde{N}_2(\sigma))} \pi^*(N, \sigma), \text{ if } 1 + 2\frac{\sigma^2}{\sigma_v^2} \notin \mathbb{N},
\]

and $\tilde{N}_1(\sigma) = \text{int} \left( 1 + 2\frac{\sigma^2}{\sigma_v^2} \right)$,

\[
\tilde{N}_2(\sigma) = \tilde{N}_1(\sigma) + 1,
\]

then in equilibrium, the aggregate profit $\pi^*(N, \sigma)$ is maximized for the value $N = N^*(\sigma)$.

**Proof:** For each given level of noise in the signals $\sigma$, the aggregate profit $\pi^*(\sigma, N)$ is maximized for the value $\tilde{N}(\sigma) = 1 + 2\frac{\sigma^2}{\sigma_v^2}$. Therefore if taking into account the possible outcome for $\tilde{N}(\sigma)$ as an integer or not, the result is obtained.

Proposition 5.3 states that for a given level of available precision $\sigma$, there exists an optimal size of the market $N^*(\sigma)$, for which the aggregate profit is maximized. Moreover, the bigger the level of noise in the signals the bigger the market. One of the empirical implications is that we
should observe a small number of informed traders in markets where the information is very sharp and conversely a larger number of insiders in market with less precise information. On the other hand for the case of direct sale of information as in Admati and Pfleiderer (1986-1990) with perfect competition, the seller should choose \( N^*(\sigma) \) participants.\(^{12}\)

6 Concluding Remarks

The main messages of this paper are threefold.

- The general study of \( N \) informed traders endowed with noisy signals and competing à la Nash in a market was still an issue that we have solved. We have shown the existence and uniqueness of a linear equilibrium.
- We have exhibited and delineated regions for which each informed traders is better off with respect to the perfectly informed case. The aggregate profit has a finite (strictly) positive limit when \( N \) is large. Therefore, even when an infinite number of insiders compete in the market the price is no longer efficient.
- For the particular situation in which each informed agent is endowed with a signal whose precision is the same, we have shown the existence of an optimal level of noise in the signals maximizing each individual profit. In the latter case, the liquidity is an increasing function of the number of informed traders but has a finite limit for large \( N \) and the informativeness of prices is a decreasing function of the number of informed traders. We also stress that for each given level of precision there exists an optimal size \( N^*(\sigma) \) of the market that maximizes the aggregate expected profit.

\(^{12}\)It is worth noticing that we do not propose in this paper any mechanism to implement the optimal noise in the signals as in Germain (1998). Anyway, the optimal noise could be reached for example if the traders collude.
References


Appendices
A.1. Proof of propositions 3.1

The agent maximizes his conditional expected profit:

$$\max_{x \in \mathbb{R}} E \left\{ x \left[ \tilde{v} - \lambda (x + \tilde{u}) \right] / S \right\}.$$ 

Since $\tilde{u} \perp \perp \tilde{S}$, we have:

$$\iff \max_{x \in \mathbb{R}} E \left\{ x \left[ \tilde{v} - \lambda x \right] / S \right\},$$

$$\iff \max_{x \in \mathbb{R}} x \left[ \frac{S}{1 + \tau} - \lambda x \right].$$

The necessary and sufficient first order conditions give:

$$\tilde{x} = \frac{1}{2\lambda} \frac{1}{1 + \tau} S = \beta S.$$ 

Thus, the informed agent’s best response is linear in his signal. Given that, at equilibrium, $\tilde{p} = E[\tilde{v}/w]$, we have thanks to the aforementioned linearity in the trading strategy $\tilde{x}$ and the normality assumption that $\tilde{p} = \lambda w$ is fulfilled and:

$$\lambda = \frac{\text{Cov}(\tilde{v}, \tilde{w})}{\text{Var}(\tilde{w})} = \frac{\text{Cov}(\tilde{v}, \beta \tilde{S} + \tilde{u})}{\text{Var}(\beta \tilde{S} + \tilde{u})} = \frac{\beta \sigma_v^2}{\beta^2 (\sigma_v^2 + \sigma_u^2) + \sigma_u^2},$$

$$\frac{1}{\lambda} = \beta (1 + \tau) + \frac{\sigma_u^2}{\sigma_v^2} \frac{1}{\beta}.$$ 

We thus have:

- $\frac{1}{\lambda} = \beta (1 + \tau) + \frac{\sigma_u^2}{\sigma_v^2} \frac{1}{\beta},$
- $\frac{1}{\lambda} = 2\beta (1 + \tau).$

We easily deduce that:

- $\beta^*(\sigma) = \frac{\sigma_u}{\sigma_v \sqrt{1 + \tau}},$
- $\frac{1}{\lambda^*(\sigma)} = 2\frac{\sigma_u}{\sigma_v \sqrt{1 + \tau}}.$

The expected profit $\pi^*(\sigma)$ is given by:

$$\pi^*(\sigma) = E \left\{ \beta^*(\sigma) \tilde{S} \left[ \tilde{v} - \lambda^*(\sigma) \tilde{w} \right] \right\},$$

$$\pi^*(\sigma) = \beta^*(\sigma) \sigma_v - \beta^{*2}(\sigma) \lambda^*(\sigma) (\sigma_v^2 + \sigma_u^2),$$

$$\pi^*(\sigma) = \sigma_u \sigma_v \frac{1}{\sqrt{1 + \tau}} - \frac{1}{2 (1 + \tau)} (1 + \tau) \frac{1}{\sqrt{1 + \tau}} \sigma_u \sigma_v,$$

$$\pi^*(\sigma) = \frac{1}{2} \sigma_u \sigma_v \frac{1}{\sqrt{1 + \tau}}.$$
This ends the proof of proposition 3.1

A.2. Proof of propositions 3.2

Agent $i$ maximizes his conditional expected profit ($N \geq 2$):

$$\max_{x \in \mathbb{R}} E \left\{ x \left[ \tilde{v} - \lambda \left( x + \sum_{j \neq i} \tilde{x}_j + \bar{u} \right) \right] / S_i \right\}, \; i = 1, \ldots, N,$$

since $\tilde{u} \perp \perp \tilde{S}_i$, we have:

$$\iff \max_{x \in \mathbb{R}} E \left\{ x \left[ \tilde{v} - \lambda \left( x + \sum_{j \neq i} \tilde{x}_j \right) \right] / S_i \right\}, \; i = 1, \ldots, N,$$

$$\max_{x \in \mathbb{R}} \frac{1}{1 + \tau_i} S_i x - \lambda E (\tilde{x}_j / S_i), \; i = 1, \ldots, N,$$

The necessary and sufficient first order conditions are:

$$\frac{1}{1 + \tau_i} S_i - 2\lambda \tilde{x}_i - \lambda \sum_{j \neq i} E (\tilde{x}_j / S_i) = 0, \; i = 1, \ldots, N,$$

$$\iff \tilde{x}_i = \frac{1}{2\lambda} \frac{1}{1 + \tau_i} S_i - \frac{1}{2} \sum_{j \neq i} E (\tilde{x}_j / S_i), \; i = 1, \ldots, N.$$ (6.1)

Let $\{x_i^*, \; i = 1, \ldots, N\}$ be a particular solution to (6.1). Then if we define:

$$\tilde{y}_i = \tilde{x}_i - x_i^*, \; i = 1, \ldots, N,$$

$\{\tilde{x}_i, \; i = 1, \ldots, N\}$ is solution to (6.1) if and only if $\{\tilde{y}_i, \; i = 1, \ldots, N\}$ is solution to (6.2):

$$\tilde{y}_i = -\frac{1}{2} \sum_{j \neq i} E (\tilde{y}_j / S_i), \; i = 1, \ldots, N.$$ (6.2)

Remark 6.1: We will in fact show that there exists a particular linear solution $x_i^* = \beta_i S_i, \; i = 1, \ldots, N$.

Remark 6.2: We will show that the set of solution (6.2) is reduced to the singleton $0 \in \mathbb{R}^N$, that is, $\{\tilde{y}_i, \; i = 1, \ldots, N\}$ is solution to (6.2) if and only if $\tilde{y}_i \equiv 0, \; i = 1, \ldots, N$. 

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If we plug the latter relation into (6.2) we have:

\[
\{ \bar{y}_i + \frac{1}{2} \sum_{j \neq i} \bar{y}_j = \bar{\eta}_i, \}
\]

with \( \bar{\eta}_i \perp \sigma(S_i) \iff E(\bar{\eta}_i/S_i) = 0 \).

We introduce:

\[
\eta \equiv (\bar{\eta}_1, \ldots, \bar{\eta}_N)^t \quad \text{and} \quad \bar{\eta} = (\bar{\eta}_1, \ldots, \bar{\eta}_N)^t.
\]

(6.2) \iff \Sigma \bar{y} = \bar{\eta}, \quad \text{that is} \quad \bar{y} = \Sigma^{-1} \bar{\eta}.

Therefore, for \( i = 1, \ldots, N \), we have:

\[
\bar{y}_i = \frac{2}{N+1} \left\{ N\bar{\eta}_i - \sum_{j \neq i} \bar{\eta}_j \right\}.
\]

If we plug the latter relation into (6.2), we obtain \( \bar{y} \) is solution to (6.2) if and only if there exist \( \{\bar{\eta}_i, \ i = 1, \ldots, N\} \) such that \( \bar{\eta}_i \perp \sigma(S_i) \) and:

(1) \( \bar{y}_i = \frac{2}{N+1} \left\{ N\bar{\eta}_i - \sum_{j \neq i} \bar{\eta}_j \right\}, \quad i = 1, \ldots, N, \)

\[
(2) \quad \frac{2}{N+1} \left\{ N\bar{\eta}_i - \sum_{j \neq i} \bar{\eta}_j \right\} = -\frac{1}{2} \sum_{j \neq i} E \left\{ \frac{2}{N+1} \left[ N\bar{\eta}_j - \sum_{k \neq j} \bar{\eta}_k \right] / S_i \right\},
\]

\( \iff \) (1) and \( N\bar{\eta}_i - \sum_{j \neq i} \bar{\eta}_j = -\frac{1}{2} \sum_{j \neq i} \sum_{k \neq j} E \left\{ N\bar{\eta}_j - \sum_{k \neq j} \bar{\eta}_k / S_i \right\}, \)

\( \iff \) (1) and \( N\bar{\eta}_i - \sum_{j \neq i} \bar{\eta}_j = -\frac{1}{2} \sum_{j \neq i} \sum_{k \neq j} E \left\{ N\bar{\eta}_j - (N-1)\bar{\eta}_i - (N-2) \sum_{k \neq i} \bar{\eta}_k / S_i \right\}, \)

\[
\sum_{j \neq i} \sum_{k \neq j} E \left\{ N\bar{\eta}_j - (N-1)\bar{\eta}_i - (N-2) \sum_{k \neq i} \bar{\eta}_k / S_i \right\} = 0.
\]

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Indeed, \( \sum_{j \neq i} \sum_{k \neq j} \tilde{\eta}_k = \sum_{j \neq i} \left[ \sum_{k=1}^{N} \tilde{\eta}_k - \tilde{\eta}_j \right] = (N-1) \sum_{k=1}^{N} \tilde{\eta}_k - \sum_{j \neq i} \tilde{\eta}_j = (N-1) \tilde{\eta}_i + (N-1) \sum_{k \neq i} \tilde{\eta}_k = (N-1) \tilde{\eta}_i + (N-2) \sum_{k \neq i} \tilde{\eta}_k \) (\( N \geq 2 \)). Thus \( \tilde{\eta}_i \) is solution to (6.2) if and only if there exist \( \{ \tilde{\eta}_i, \ i = 1, \ldots, N \} \) such that \( \tilde{\eta}_i \perp \sigma(S_i) \) and:

\[
\tilde{\eta}_i = \frac{2}{N+1} \left\{ \frac{N \tilde{\eta}_i - \sum_{j \neq i} \tilde{\eta}_j}{N} \right\}, \quad i = 1, \ldots, N,
\]

\[
N \tilde{\eta}_i - \sum_{j \neq i} \tilde{\eta}_j = - E \left[ \sum_{j \neq i} \tilde{\eta}_j / S_i \right],
\]

since \( E [\tilde{\eta}_i / S_i] = 0 \). We have \( N \geq 2 \):

\[
N \tilde{\eta}_i = \sum_{j \neq i} \tilde{\eta}_j - E \left[ \sum_{j \neq i} \tilde{\eta}_j / S_i \right], \quad i = 1, \ldots, N.
\]

From the latter equation, we deduce that \( E (\tilde{\eta}_i) = 0, \ i = 1, \ldots, N \) since \( E (\tilde{\eta}_i) = \frac{1}{N} E \{ \bar{z} - E [\bar{z} / S_i] \} = 0 \) where \( \bar{z} = \sum_{j \neq i} \tilde{\eta}_j \).

\[
Var (\tilde{\eta}_i) = \frac{1}{N^2} Var \{ \bar{z} - E [\bar{z} / S_i] \} = \frac{1}{N^2} E [Var (\bar{z} / S_i)],
\]

\[
\leq \frac{1}{N^2} Var (\bar{z}) = \frac{1}{N^2} Var \left( \sum_{j \neq i} \tilde{\eta}_j \right) \leq \frac{1}{N^2} \left[ \sum_{j \neq i} \sqrt{Var (\tilde{\eta}_j)} \right]^2,
\]

Indeed

\[
Var \left( \sum_{j \neq i} \tilde{\eta}_j \right) = \sum_{j \neq i} Var (\tilde{\eta}_j) + 2 \sum_{j, k \neq i, k < j} Cov (\tilde{\eta}_j, \tilde{\eta}_k) \leq \sum_{j \neq i} \left[ \sqrt{Var (\tilde{\eta}_j)} \right]^2 + 2 \sum_{j, k \neq i, k < j} \sqrt{Var (\tilde{\eta}_j)} \sqrt{Var (\tilde{\eta}_k)} = \left[ \sum_{j \neq i} \sqrt{Var (\tilde{\eta}_j)} \right]^2.
\]

\[
\Rightarrow \forall i = 1, \ldots, N, \ Var (\tilde{\eta}_i) \leq \left( \frac{N-1}{N} \right)^2 \max_{j=1, \ldots, N} \{ Var (\tilde{\eta}_j) \},
\]

\[
\Rightarrow \max_{i=1, \ldots, N} \ Var (\tilde{\eta}_i) \leq \left( \frac{N-1}{N} \right)^2 \max_{j=1, \ldots, N} \{ Var (\tilde{\eta}_j) \},
\]

\[
\Rightarrow \left[ 1 - \left( 1 - \frac{1}{N} \right)^2 \right] \max_{i=1, \ldots, N} \ Var (\tilde{\eta}_i) \leq 0,
\]

\[
\Rightarrow \max_{i=1, \ldots, N} \ Var (\tilde{\eta}_i) = 0.
\]
Thus we have for $i = 1, \ldots, N$:

\[
\begin{align*}
E(\tilde{\eta}_i) &= 0, \\
\text{Var}(\tilde{\eta}_i) &= 0,
\end{align*}
\]

\[\implies \tilde{\eta}_i \text{ a.s.} = 0.\]

Therefore the only solution to (6.2) is $\tilde{y} = 0 \in \mathbb{R}^N$.

We now show the existence of a linear equilibrium $\{x_i^*, i = 1, \ldots, N\}$ which is, in addition, linear in the information. Hence and from the previous property, it corresponds to the unique linear equilibrium. We thus look for a particular solution to (6.2) of the form: $\{x_i^* = \beta_i^* S_i, i = 1, \ldots, N\}$. We introduce the following quantities:

\[
a_i = \frac{1}{1 + \tau_i} < 1, \quad i = 1, \ldots, N,
\]

\[
b_i = \frac{1}{2\lambda a_i}.
\]

Thanks to the normality assumption and the mutual independence of $\{\tilde{v}, \tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_N\}$, we have

\[
E(\tilde{S}_j/S_i) = a_i S_i \text{ for } j \neq i.
\]

Using (6.1), we have:

\[
\beta_i S_i = b_i S_i - \frac{1}{2} \sum_{j \neq i} E[\beta_j \tilde{S}_j/S_i], \quad i = 1, \ldots, N,
\]

or equivalently:

\[
\beta_i = b_i - \frac{a_i}{2} \sum_{j \neq i} \beta_j, \quad i = 1, \ldots, N;
\]

\[
\beta_i \left(1 - \frac{a_i}{2}\right) = b_i - \frac{a_i}{2} \sum_{j=1}^N \beta_j, \quad i = 1, \ldots, N;
\]

\[
\beta_i \left(\frac{2 - a_i}{2a_i}\right) = \frac{1}{2\lambda} - \frac{1}{2} \sum_{j=1}^N \beta_j, \quad i = 1, \ldots, N;
\]

\[
\beta_i \left(\frac{1}{2} + \tau_i\right) = \frac{1}{2\lambda} - \frac{1}{2} \sum_{j=1}^N \beta_j, \quad i = 1, \ldots, N,
\]

\[
\beta_i = \frac{a}{1 + 2\tau_i},
\]

\[\iff \quad \frac{1}{\lambda} = a \left[1 + \sum_{j=1}^N \frac{1}{1 + 2\tau_j}\right], \quad \text{(6.3)}\]

with $a$ independent of $i$. We also know that, at equilibrium, $\tilde{p} = E(\tilde{v}/w)$. In this particular equilibrium, we have $\tilde{w} = \sum_{i=1}^N \beta_i \tilde{S}_i + \tilde{u}$. Thanks to the mutual independence of $\{\tilde{v}, \tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_N, \tilde{u}\}$
and normality of each components, we deduce that \( \bar{p} = \lambda w \) holds and that therefore we do obtain a linear equilibrium with:

\[
\lambda = \frac{\text{Cov} (\tilde{v}, \tilde{w})}{\text{Var} (\tilde{w})} = \frac{\text{Cov} \left( \tilde{v}, \sum_{j=1}^{N} \beta_j \tilde{S}_j + \tilde{u} \right)}{\text{Var} \left( \sum_{j=1}^{N} \beta_j \tilde{S}_j + \tilde{u} \right)},
\]

\[
\frac{N}{\text{Var} \left( \sum_{j=1}^{N} \beta_j \tilde{v} + \sum_{j=1}^{N} \beta_j \tilde{S}_j + \tilde{u} \right)} = \frac{N}{\sum_{j=1}^{N} \beta_j \sigma_v^2 + \sum_{j=1}^{N} \beta_j^2 \sigma_S^2 + \sigma_u^2},
\]

\[
\frac{1}{\lambda} = a \sum_{j=1}^{N} \frac{1}{1 + 2 \tau_j} + \frac{\sum_{j=1}^{N} \frac{1}{1 + 2 \tau_j}}{a \sum_{j=1}^{N} \frac{1}{1 + 2 \tau_j}} + \frac{\frac{\sigma_u^2}{\sigma_v^2}}{a \sum_{j=1}^{N} \frac{1}{1 + 2 \tau_j}}.
\]

Using (6.3) and (6.4), we obtain:

\[
a \left\{ \frac{\sum_{j=1}^{N} \frac{\tau_j}{(1 + 2 \tau_j)^2}}{\sum_{j=1}^{N} \frac{1}{1 + 2 \tau_j}} \right\} = \frac{\frac{\sigma_u^2}{\sigma_v^2}}{a \sum_{j=1}^{N} \frac{1}{1 + 2 \tau_j}},
\]

\[
a^2 \left\{ \frac{\sum_{j=1}^{N} \frac{1}{1 + 2 \tau_j} - \sum_{j=1}^{N} \frac{\tau_j}{(1 + 2 \tau_j)^2}}{a \sum_{j=1}^{N} \frac{1}{1 + 2 \tau_j}} \right\} = \frac{\frac{\sigma_u^2}{\sigma_v^2}}{a \sum_{j=1}^{N} \frac{1}{1 + 2 \tau_j}},
\]

\[
a^2 \sum_{j=1}^{N} \frac{1 + \tau_j}{(1 + 2 \tau_j)^2} = \frac{\frac{\sigma_u^2}{\sigma_v^2}}{a \sum_{j=1}^{N} \frac{1}{1 + 2 \tau_j}}.
\]
\[
a = \frac{\sigma_u}{\sigma_v} \left\{ \frac{1}{\sum_{j=1}^{N} \frac{1 + \tau_j}{(1 + 2\tau_j)^2}} \right\}^{\frac{1}{2}} \\

\beta^*_i(\tau) = \frac{\sigma_u}{\sigma_v} \frac{1}{1 + 2\tau_i} \left\{ \frac{1}{\sum_{j=1}^{N} \frac{1 + \tau_j}{(1 + 2\tau_j)^2}} \right\}^{\frac{1}{2}} \\

\frac{1}{\lambda^*(\tau)} = \frac{\sigma_u}{\sigma_v} \left\{ \frac{1}{\sum_{j=1}^{N} \frac{1 + \tau_j}{(1 + 2\tau_j)^2}} \right\}^{\frac{1}{2}} \left[ 1 + \sum_{j=1}^{N} \frac{1}{1 + 2\tau_j} \right].
\]

\[
\pi^*_i(\sigma) = \beta^*_i \left( 1 - \lambda^* \sum_{j=1}^{N} \beta^*_j \right) \sigma_v^2 - \lambda^* \beta^*_i \sigma_v^2 = \sigma_v^2 \left[ \beta^*_i \left( 1 - \lambda^* \sum_{j=1}^{N} \beta^*_j \right) - \lambda^* \beta^*_i \tau_i \right],
\]

\[
\beta^*_i(\sigma) = \frac{a}{1 + 2\tau_i}, \quad a = \frac{\sigma_u}{\sigma_v} \left\{ \frac{1}{\sum_{j=1}^{N} \frac{1 + \tau_j}{(1 + 2\tau_j)^2}} \right\}^{\frac{1}{2}}
\]

\[
\lambda^*(\sigma) = \frac{1}{a} \left( 1 + \sum_{j=1}^{N} \frac{1}{1 + 2\tau_j} \right), \quad \lambda^*(\sigma) \beta^*_i(\sigma) = \frac{1}{1 + 2\tau_i} \left( 1 + \sum_{j=1}^{N} \frac{1}{1 + 2\tau_j} \right)
\]

\[
\sum_{j=1}^{N} \lambda^* \beta^*_i(\sigma) = 1 - \frac{1}{1 + \sum_{j=1}^{N} \frac{1}{1 + 2\tau_j}}
\]

\[
\pi^*_i(\sigma) = \sigma_v^2 \left\{ \frac{a}{1 + 2\tau_i} \left( 1 + \sum_{j=1}^{N} \frac{1}{1 + 2\tau_j} \right) \right\}^{\frac{1}{2}} - \frac{a}{(1 + 2\tau_i)^2} \left( 1 + \sum_{j=1}^{N} \frac{1}{1 + 2\tau_j} \right)
\]

\[
\pi^*_i(\sigma) = \frac{\sigma_v^2 a}{(1 + 2\tau_i) \left( 1 + \sum_{j=1}^{N} \frac{1}{1 + 2\tau_j} \right)} \left\{ 1 - \frac{\tau_i}{1 + 2\tau_i} \right\} = \frac{\sigma_v^2 a}{(1 + 2\tau_i) \left( 1 + \sum_{j=1}^{N} \frac{1}{1 + 2\tau_j} \right)} \left( 1 + \sum_{j=1}^{N} \frac{1}{1 + 2\tau_j} \right)
\]

\[
\pi^*_i(\sigma) = \frac{\sigma_v \sigma_u}{1 + \sum_{j=1}^{N} \frac{1}{1 + 2\tau_j}} \left\{ \frac{\sum_{j=1}^{N} \frac{1 + \tau_j}{(1 + 2\tau_j)^2}}{\frac{1}{4} [1 + 2\tau_i]^2} \right\}
\]

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The aggregate profit is straightforwardly deduced. This ends the proof of proposition 3.2.

B.1. Proof of proposition 4.1

Indeed:

\[
\frac{\partial \beta^*_i}{\partial \tau_j}(\tau) = \frac{1}{2} \frac{\sigma_u}{\sigma_v} \left\{ \sum_{j=1}^{N} \frac{1 + \tau_j}{(1 + 2 \tau_j)^{3/2}} \right\} \left( 1 + 2 \tau_i \right)^{3/2} > 0, \quad \text{for } j \neq i.
\]

\[
\frac{\partial \beta^*_i}{\partial \tau_i}(\tau) = \frac{1}{2} \frac{\sigma_u}{\sigma_v} \left\{ \sum_{j=1}^{N} \frac{1 + \tau_j}{(1 + 2 \tau_j)^{3/2}} \right\} \left( 1 + 2 \tau_i \right)^{3/2} < 0.
\]

Indeed:

\[
\frac{\partial \beta^*_i}{\partial \tau_i}(\tau) = \frac{1}{2} \frac{\sigma_u}{\sigma_v} \left\{ \sum_{j=1}^{N} \frac{1 + \tau_j}{(1 + 2 \tau_j)^{3/2}} \right\} \left( 1 + 2 \tau_i \right)^{3/2} - \frac{\sigma_u}{\sigma_v} \left\{ \sum_{j=1}^{N} \frac{1 + \tau_j}{(1 + 2 \tau_j)^{3/2}} \right\} \left( 1 + 2 \tau_i \right)^{3/2},
\]

\[
\frac{\partial \beta^*_i}{\partial \tau_i}(\tau) = \frac{1}{2} \frac{\sigma_u}{\sigma_v} \left\{ \sum_{j=1}^{N} \frac{1 + \tau_j}{(1 + 2 \tau_j)^{3/2}} \right\} \left( 1 + 2 \tau_i \right)^{3/2} - \frac{\sigma_u}{\sigma_v} \left\{ \sum_{j=1}^{N} \frac{1 + \tau_j}{(1 + 2 \tau_j)^{3/2}} \right\} \left( 1 + 2 \tau_i \right)^{3/2},
\]

\[
\frac{\partial \ln \pi^*_i}{\partial \tau_j}(\tau) = \frac{\partial}{\partial \tau_j} \left\{ -\frac{1}{2} \ln \left[ \sum_{k=1}^{N} \frac{1 + \tau_k}{(1 + 2 \tau_k)^{3/2}} \right] - \ln \left[ 1 + \sum_{k=1}^{N} \frac{1}{1 + 2 \tau_k} \right] \right\},
\]

\[
= \frac{1}{2} \sum_{k=1}^{N} \frac{1 + \tau_k}{(1 + 2 \tau_k)^{3/2}} \left( 1 + 2 \tau_j \right)^{3/2} + \frac{2}{1 + \sum_{k=1}^{N} \frac{1}{1 + 2 \tau_k}} \left( 1 + 2 \tau_j \right)^{3/2} > 0,
\]

\[\implies \pi^*_i(\sigma) \text{ is increasing in } \sigma_{-i}.
\]
This ends the proof of proposition 4.

\[
\frac{\partial \ln \pi_\ast_i}{\partial \tau_i}(\tau) = \frac{1}{2} \left( \frac{3 + 2\tau_i}{(1 + 2\tau_i)^3} \sum_{k=1}^{N} \frac{1}{1 + 2\tau_k} \right) + \frac{2}{(1 + 2\tau_i)^2} \left[ \sum_{k=1}^{N} \frac{1}{1 + 2\tau_k} \right] + \frac{1}{1 + \tau_i} - \frac{4}{1 + 2\tau_i},
\]

\[
= \frac{1}{2} \frac{3 + 2\tau_i}{(1 + 2\tau_i)^3} \sum_{k=1}^{N} \frac{1}{1 + 2\tau_k} + \frac{2}{(1 + 2\tau_i)^2} \left[ \sum_{k=1}^{N} \frac{1}{1 + 2\tau_k} \right] - \frac{3 + 2\tau_i}{(1 + 2\tau_i)(1 + \tau_i)},
\]

\[
= \frac{1}{2} \frac{3 + 2\tau_i}{(1 + 2\tau_i)^3} \sum_{k=1}^{N} \frac{1}{1 + 2\tau_k} \left[ \sum_{k=1}^{N} \frac{1}{1 + 2\tau_k} \right] (1 + \tau_i)
\]

where:

\[
h_i(\tau) = (3 + 2\tau_i)(1 + \tau_i) \left[ 1 + \sum_{k=1}^{N} \frac{1}{1 + 2\tau_k} \right] + 4(1 + 2\tau_i) \left[ \sum_{k=1}^{N} \frac{1 + \tau_k}{(1 + 2\tau_k)^2} \right] (1 + \tau_i)
\]

\[
= (3 + 2\tau_i)(1 + \tau_i) + \frac{(3 + 2\tau_i)(1 + \tau_i)}{1 + 2\tau_i} + (3 + 2\tau_i)(1 + \tau_i) \sum_{k=1}^{N} \frac{1}{1 + 2\tau_k} + 4 \frac{(1 + \tau_i)^2}{1 + 2\tau_i}
\]

\[
+ 4(1 + 2\tau_i)(1 + \tau_i) \sum_{k=1}^{N} \frac{1 + \tau_k}{(1 + 2\tau_k)^2} - 2(3 + 2\tau_i)(1 + \tau_i) - 2(3 + 2\tau_i)(1 + 2\tau_i) \sum_{k=1}^{N} \frac{1 + \tau_k}{(1 + 2\tau_k)^2}
\]

\[
- \frac{2(3 + 2\tau_i)(1 + \tau_i)}{1 + 2\tau_i} - 2(3 + 2\tau_i)(1 + \tau_i) \sum_{k=1}^{N} \frac{1}{1 + 2\tau_k} - 2(3 + 2\tau_i)(1 + 2\tau_i) \sum_{k=1}^{N} \frac{1 + \tau_k}{(1 + 2\tau_k)^2}
\]

\[
- 2(3 + 2\tau_i)(1 + 2\tau_i)^2 \sum_{k=1}^{N} \frac{1 + \tau_k}{(1 + 2\tau_k)^2} \sum_{k=1}^{N} \frac{1}{1 + 2\tau_k}
\]

\[
= -2(1 + \tau_i)^2 - (3 + 2\tau_i)(1 + \tau_i) \sum_{k=1}^{N} \frac{1}{1 + 2\tau_k} - 8(1 + 2\tau_i)(1 + \tau_i)^2 \sum_{k=1}^{N} \frac{1 + \tau_k}{(1 + 2\tau_k)^2}
\]

\[
- 2(3 + 2\tau_i)(1 + 2\tau_i)^2 \sum_{k=1}^{N} \frac{1 + \tau_k}{(1 + 2\tau_k)^2} \sum_{k=1}^{N} \frac{1}{1 + 2\tau_k}
\]

\[h_i(\tau) < 0 \implies \pi_\ast_i(\sigma) \text{ is decreasing in } \tau_i = \frac{\sigma_i^2}{\sigma_v^2}.
\]

This ends the proof of proposition 4.1.

**B.2. Proof of proposition 4.2**

\[
\frac{\partial \lambda^{-1}}{\partial \tau_i}(\tau) = \frac{\sigma_u}{\sigma_v} \left\{ \frac{1}{2} \left[ \frac{(3 + 2\tau_i)(1 + \sum_{j=1}^{N} \frac{1}{1 + 2\tau_j})}{\left( \sum_{j=1}^{N} \frac{1 + \tau_j}{(1 + 2\tau_j)^2} \right)^{\frac{3}{2}}} \right] - \frac{2}{\left( \sum_{j=1}^{N} \frac{1 + \tau_j}{(1 + 2\tau_j)^2} \right)^{\frac{1}{2}}} \right\},
\]

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In order to show the result of the non-emptiness of $\Sigma_N$ for $N \geq 4$, we just need to focus on the symmetric case where $\sigma_1 = \ldots = \sigma_N = \sigma$ and this is the purpose of proposition 5.1. We now show that $\Sigma_N$ is empty for $N \leq 3$.

**Proof for $N=2$**

$$
\pi_i(\tau) = \sigma_u \sigma_v \frac{1 + \tau_i}{(1 + 2\tau_i)^2} \cdot \left\{ \sum_{j=1}^{2} \frac{1 + \tau_j}{(1 + 2\tau_j)^2} \right\}^{\frac{3}{2}} \cdot \left\{ 1 + \sum_{i=1}^{2} \frac{1}{1 + 2\tau_i} \right\}.
$$

**B.3. Proof of proposition 4.3**

We define $b(\tau_{-i}) = 1 - \sum_{j \neq i} \frac{3 + 2\tau_j}{(1 + 2\tau_j)^2}$, $c(\tau_{-i}) = 2 + \sum_{j \neq i} \frac{2\tau_j - 1}{(1 + 2\tau_j)^2}$. It is worth noticing that $c(\tau_{-i}) > b(\tau_{-i})$, indeed $c(\tau_{-i}) - b(\tau_{-i}) = 1 + \sum_{j \neq i} \frac{2}{1 + 2\tau_j}$. This ends the proof of proposition 4.2.

$$
\frac{\partial \lambda^{-1}}{\partial \tau_i}(\tau) = \frac{1}{2} \left\{ \sum_{j=1}^{N} \frac{1 + \tau_j}{(1 + 2\tau_j)^2} \right\}^{\frac{3}{2}} \cdot (1 + 2\tau_i)^3
$$

with:

$$
h_i(\tau) = 3 + 2\tau_i + \left( 3 + 2\tau_i \right) \sum_{j \neq i} \frac{1}{1 + 2\tau_j} - 4 \left( 1 + 2\tau_i \right) \sum_{j \neq i} \frac{1 + \tau_j}{(1 + 2\tau_j)^2},
$$

$$
h_i(\tau) = 2 + 2\tau_i + \left( 3 + 2\tau_i \right) \sum_{j \neq i} \frac{1}{1 + 2\tau_j} - 4 \left( 1 + 2\tau_i \right) \sum_{j \neq i} \frac{1 + \tau_j}{(1 + 2\tau_j)^2},
$$

$$
h_i(\tau) = \tau_i \left( 2 + 2 \sum_{j \neq i} \frac{1}{1 + 2\tau_j} - 8 \sum_{j \neq i} \frac{1 + \tau_j}{(1 + 2\tau_j)^2} \right) + 2 + 3 \sum_{j \neq i} \frac{1}{1 + 2\tau_j} - 4 \sum_{j \neq i} \frac{1 + \tau_j}{(1 + 2\tau_j)^2},
$$

$$
h_i(\tau) = 2\tau_i \left[ 1 - \sum_{j \neq i} \frac{3 + 2\tau_j}{(1 + 2\tau_j)^2} \right] + 2 + \sum_{j \neq i} \frac{2\tau_j - 1}{(1 + 2\tau_j)^2}.
$$

We define $b(\tau_{-i}) = 1 - \sum_{j \neq i} \frac{3 + 2\tau_j}{(1 + 2\tau_j)^2}$, $c(\tau_{-i}) = 2 + \sum_{j \neq i} \frac{2\tau_j - 1}{(1 + 2\tau_j)^2}$. It is worth noticing that $c(\tau_{-i}) > b(\tau_{-i})$, indeed $c(\tau_{-i}) - b(\tau_{-i}) = 1 + \sum_{j \neq i} \frac{2}{1 + 2\tau_j}$. This ends the proof of proposition 4.2.
We denote $x_i = \frac{1 + \tau_i}{(1 + 2\tau_i)^2}$, $i = 1, 2$. For $\tau_i \neq 0$, $x_i < 1$ since $x_i = \frac{1}{1 + 2\tau_i} - \frac{\tau_i}{(1 + 2\tau_i)^2}$.

$y_i = \frac{1}{1 + 2\tau_i} = \frac{\sqrt{1 + 8x_i} - 1}{2}$ indeed:

$$x_i = \frac{1}{2} y_i + \frac{1}{2} y_i^2 \implies y_i = \frac{\sqrt{1 + 8x_i} - 1}{2} > 0.$$ 

Therefore we have $i = 1, 2$:

$$\pi_i(\tau) = \frac{\sigma_u \sigma_v}{3\sqrt{2}} \cdot \frac{2x_i}{\left(\sum_{j=1}^{2} x_j\right) \left(\sum_{j=1}^{2} \sqrt{1 + 8x_j}\right)}.$$ 

$$\pi_i(0) = \frac{\sigma_u \sigma_v}{3\sqrt{2}}.$$ 

We already know that, for $\tau_1 = \tau_2 \neq 0$, $\pi_1(\tau)$ and $\pi_2(\tau)$ are strictly smaller than $\pi_i(0) = \frac{\sigma_u \sigma_v}{3\sqrt{2}}$ and equal if and only if $\tau_1 = \tau_2 = 0$. We thus focus on cases where $\tau_1 \neq \tau_2 \iff x_1 \neq x_2$.

Suppose now (without any loss of generality) that $x_1 > x_2$ and that $\pi_i(\tau) \geq \frac{\sigma_u \sigma_v}{3\sqrt{2}}$, $i = 1, 2$.

$$\implies x_i \geq \frac{1}{6\sqrt{2}} \left(\sum_{j=1}^{2} x_j\right)^{\frac{1}{2}} \left(\sum_{j=1}^{2} \sqrt{1 + 8x_j}\right), \ i = 1, 2,$$

since $x_1 > x_2$:

$$\implies x_2 \geq \frac{1}{3} \sqrt{x_2} \sqrt{1 + 8x_2},$$

$$\implies \sqrt{x_2} \geq \frac{1}{3} \sqrt{1 + 8x_2},$$

$$\implies x_2 \geq \frac{1}{9} \left(1 + 8x_2\right),$$

$$\implies x_2 \geq 1,$$

which is ruled out. This implies that $\Sigma_2$ is empty.

**Proof for N=3**

$$\pi_i(\tau) = \frac{\sigma_u \sigma_v}{3\sqrt{2}} \cdot \frac{1 + \tau_i}{(1 + 2\tau_i)^2} \cdot \left\{\sum_{j=1}^{3} \frac{1 + \tau_j}{(1 + 2\tau_j)^2}\right\}^{\frac{1}{2}} \left\{1 + \sum_{i=1}^{3} \frac{1}{1 + 2\tau_i}\right\}.$$
We again denote \( x_i = \frac{1 + \tau_i}{(1 + 2\tau_i)^2} \), \( i = 1, 2, 3 \). For \( \tau_i \neq 0 \), \( x_i < 1 \), \( y_i = \frac{1}{1 + 2\tau_i} = \frac{\sqrt{1 + 8x_i} - 1}{2} \).

Therefore we have \( i = 1, 2, 3 \):

\[
\pi_i(\tau) = \sigma_v\sigma_u \frac{2x_i}{\left( \sum_{j=1}^{3} x_j \right)^{\frac{1}{2}} \left( \frac{3}{3} \sqrt{1 + 8x_j - 1} \right)},
\]

\[
\pi_i(0) = \frac{\sigma_v\sigma_u}{4\sqrt{3}}.
\]

We already know that, for \( \tau_1 = \tau_2 = \tau_3 \neq 0 \), \( \pi_1(\tau) \), \( \pi_2(\tau) \) and \( \pi_3(\tau) \) are strictly smaller than \( \pi_i(0) = \frac{\sigma_v\sigma_u}{4\sqrt{3}} \) and equal if and only if \( \tau_1 = \tau_2 = \tau_3 = 0 \). We thus focus on cases where \( \exists (i, j) \in \{1, 2, 3\}^2 \) such that \( \tau_j \neq \tau_i \iff x_j \neq x_i \). Without any loss of generality (since it corresponds to the generic case), we will suppose (up to a reordering) that:

a) either \( x_1 > x_2 \geq x_3 \),

b) or \( x_1 \geq x_2 > x_3 \).

Now assume that \( \pi_i(\tau) \geq \frac{\sigma_v\sigma_u}{4\sqrt{3}}, \ i = 1, 2, 3 \).

a) \( x_1 > x_2 \geq x_3 \):

\[
\Rightarrow x_3 \geq \frac{1}{8\sqrt{3}} \left( \sum_{j=1}^{3} x_j \right)^{\frac{1}{2}} \left( \sum_{j=1}^{3} \sqrt{1 + 8x_j - 1} \right),
\]

\[
\Rightarrow x_3 \geq \frac{1}{8\sqrt{x_3}} \left( 3\sqrt{1 + 8x_3} - 1 \right),
\]

\[
\Rightarrow 8\sqrt{x_3} \geq 3\sqrt{1 + 8x_3} - 1.
\]

Let \( f \) be:

\[
f(x) = 8\sqrt{x} - 3\sqrt{1 + 8x} + 1,
\]

\[
0 \leq x,
\]

\[
f'(x) = \frac{4(1 - x)}{\sqrt{x} \sqrt{1 + 8x} \sqrt{\sqrt{1 + 8x + 3\sqrt{x}}}}.
\]

\( f \) is increasing on \([0, 1]\), decreasing on \([1, +\infty[\) and \( f(1) = 0 \). Therefore \( \forall x \geq 0 \), \( f(x) \leq 0 \) and \( f(x) = 0 \iff x = 1 \).

\[
8\sqrt{x_3} \geq 3\sqrt{1 + 8x_3} - 1 \iff x_3 = 1,
\]

which is ruled out.

b) \( x_1 \geq x_2 > x_3 \): the same proof is available.
This implies that $\Sigma_3$ is empty.

C.1. Proof of proposition 5.1

\[ N \geq 4, \quad \pi_i^*(\sigma) > \pi_i^*(0), \]
\[ \iff \frac{\sqrt{1 + \tau}}{\sqrt{N} (N + 1 + 2\tau)} > \frac{1}{\sqrt{N} (N + 1)}, \quad \text{where} \quad \tau = \frac{\sigma^2}{\sigma_v^2}, \]
\[ \iff \sqrt{1 + \tau} > 1 + \frac{2}{N + 1} \tau, \]
\[ \iff 1 + \tau > 1 + \frac{4}{N + 1} \tau + \frac{4}{(N + 1)^2} \tau^2, \]
\[ \iff \left( 1 - \frac{4}{N + 1} \right) \tau > \frac{4}{(N + 1)^2} \tau^2, \quad \tau > 0, \]
\[ \iff \tau < \frac{(N - 3)(N + 1)}{4}, \]
\[ \iff \sigma < \frac{\sigma_v}{2} \sqrt{N - 3} \sqrt{N + 1}. \]

\[ \pi_i^*(\tau) = \sigma_v \sigma_u \frac{\sqrt{1 + \tau}}{\sqrt{N} (N + 1 + 2\tau)}, \quad \text{where} \quad \tau = \frac{\sigma^2}{\sigma_v^2}, \]
\[ \frac{\partial \ln \pi_i^*}{\partial \tau}(\tau) = \frac{\partial}{\partial \tau} \left\{ \frac{1}{2} \ln (1 + \tau) - \ln (N + 1 + 2\tau) \right\}, \]
\[ = \frac{1}{2(1 + \tau)} - \frac{2}{N + 1 + 2\tau} = \frac{N - 3 - 2\tau}{2(1 + \tau)(N + 1 + 2\tau)}. \]

Therefore $\pi_i^*(\tau)$ is maximized for $\tau = \frac{N - 3}{2}$. This ends the proof of proposition 5.1 and by the way of proposition 4.3.

C.2. Proof of proposition 5.2

\[
\begin{pmatrix}
\tilde{v} \\
\tilde{w}
\end{pmatrix}
\sim \mathcal{N}\left( \begin{pmatrix}
0 \\
0
\end{pmatrix}, \begin{pmatrix}
\sigma_v^2 & \frac{\sqrt{N} \sigma_u \sigma_v}{\sqrt{1 + \frac{\sigma^2}{\sigma_v^2}}} \\
\frac{\sqrt{N} \sigma_u \sigma_v}{\sqrt{1 + \frac{\sigma^2}{\sigma_v^2}}} & \sigma_u^2 \left[ \frac{N + \frac{\sigma^2}{\sigma_v^2}}{1 + \frac{\sigma^2}{\sigma_v^2}} \right]
\end{pmatrix} \right),
\]
\[
\begin{pmatrix}
\tilde{v} \\
\tilde{u}
\end{pmatrix}
\text{ is normal } \implies
\begin{pmatrix}
\tilde{v} \\
\tilde{u}
\end{pmatrix}
\text{ is normal. Moreover we have:}
\]

- \( \text{Cov} (\tilde{v}, \tilde{w}) = \text{Cov} \left( \tilde{v}, N\beta^* (\sigma, N)\tilde{v} + \sum_{i=1}^{N} \beta^* (\sigma, N)\tilde{e}_i + \tilde{u} \right) = N\beta^* (\sigma, N)\sigma^2_v = \frac{\sqrt{N}\sigma_u \sigma_v}{\sqrt{1 + \frac{\sigma^2}{\sigma^2_v}}} \)

- \( \text{Var} (\tilde{w}) = \text{Var} \left( N\beta^* (\sigma, N)\tilde{v} + \sum_{i=1}^{N} \beta^* (\sigma, N)\tilde{e}_i + \tilde{u} \right) = N^2 \beta^* (\sigma, N)\sigma^2_v + N\beta^* (\sigma, N)\sigma^2 + \sigma^2_u, \)

\[
\text{Var} (\tilde{w}) = \frac{N\sigma^2_u}{1 + \frac{\sigma^2}{\sigma^2_v}} + \frac{\sigma^2_u \sigma^2_v}{1 + \frac{\sigma^2}{\sigma^2_v}} + \sigma^2_u = \sigma^2_u \left[ 1 + \frac{N + \sigma^2_v}{1 + \frac{\sigma^2}{\sigma^2_v}} \right],
\]

- \( \text{Var} (\tilde{v}/w) = \text{Var} (\tilde{v}) - \left[ \frac{\text{Cov} (\tilde{v}, \tilde{w})}{\text{Var} (\tilde{w})} \right]^2 = \sigma^2_v \left[ 1 - \frac{N}{N + 1 + \frac{2\sigma^2}{\sigma^2_v}} \right] \).

The rest of proposition 5.2 is proved while using standard central limit theory and expansion theory and is therefore omitted.